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LETTER TO THE EDITOR

Universality in the two-dimensional continuous spin model

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Abstract. Monte Carlo methods have been used to analyse the critical behaviour of the $d=2$ 'border' ϕ^4 model which, according to a recent series expansion study, violates expectations based upon universality. The results suggest that the model does, in fact, fall into the Ising universality class, at least as regards those parameters which characterise the critical point configurations of the ordering variable.

In a recent paper Baker and Johnson (1984, to be referred to as BJ) have presented results setting a question mark beside the widely held view that the ϕ^4 (or 'continuous spin') model of phase transitions falls into the same universality class as the Ising ('fixed length spin') model. Specifically, high-temperature series expansions for the order-parameter susceptibility were analysed for a particular ('border model') member of the spectrum of ϕ^4 models, in space dimension $d=2$, and found to yield a susceptibility exponent γ close to 2, disturbingly different from the (exact) value appropriate to the $d=2$ Ising model, $\gamma_I = 1.75$. The inference, drawn by BJ, of a failure of universality has since been questioned by Barma and Fisher (1984) who have identified a similar effect in two related $d=2$ models (the Klauder and double-Gaussian models). These authors present convincing evidence that the effect is a manifestation of strong corrections to scaling, and that the asymptotic universal behaviour in these models is indeed Ising-type. They surmise that the behaviour of the ϕ^4 model might be explained in a similar vein. However, in a further communication, Baker and Johnson (1985) have questioned this proposition. They note, moreover, that their border model values for the indices γ and ν (the latter based upon an as yet unreported study) are consistent with one of the sets of indices (namely $\gamma=2$, $\nu=\frac{20}{19}$, $\beta/\nu=\eta/2=\frac{1}{20}$) identified by Friedan *et al* (1984) in a general study of two-dimensional systems possessing conformal invariance and unitarity. The issue remains controversial (Fisher and Barma 1985).

In this paper the issue is addressed with the aid of a Monte Carlo (MC) study of the $d=2$ border ϕ^4 model. The results, though not conclusive, support the conventional view, providing evidence that at least a subset of the (nominally universal) critical point parameters are indeed those of the Ising model.

The ϕ^4 model to be studied is defined by the partition function

$$Z=\prod_i \int \mathrm{d}\phi_i \exp(-\mathscr{H}[\{\phi\}]) \tag{1a}$$

where

$$\mathscr{H}[\{\phi\}]=\sum_i W(\phi_i)-C\sum_{\langle ij\rangle}\phi_i\phi_j \tag{1b}$$

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and

$$W(\phi) = A\phi^2 + B\phi^4. \tag{1c}$$

The sum on i extends over the L^2 sites of a $d=2$ square lattice; the sum on $\langle ij\rangle$ extends over all pairs of neighbouring sites. Thus defined the model has three parameters. Given the arbitrariness of the scale of the ϕ field one parameter is redundant and (following BJ) may be eliminated by imposing on A and B the constraint that the second moment of ϕ with respect to the weight function $e^{-W(\phi)}$ be unity. The border model, studied by BJ, is then defined by the further condition $A=0$ which, given the first condition, implies $B=0.114\,23\ldots$.

It is possible to implement a MC analysis of the ϕ^4 model in many different ways (cf Creutz and Friedman 1981, Cooper *et al* 1982). The strategy adopted here is as follows. The partition function (1*a*) is rewritten in the form

$$Z = \prod_i \int d\phi_i\, p^{(0)}(\phi_i) \exp(-\mathscr{H}'[\{\phi\}]) \tag{2a}$$

where

$$\mathscr{H}'[\{\phi\}] = \mathscr{H}[\{\phi\}] + \sum_i \ln p^{(0)}(\phi_i). \tag{2b}$$

The function $p^{(0)}(\phi)$ is chosen to be that sum of two Gaussians which best approximates the true distribution $p(\phi)$ of the local coordinate. (An initial guess as to the form of this function is subsequently refined on the basis of the computed form of $p(\phi)$.) The two-stage updating scheme for a particular coordinate ϕ_i is then: (i) A new value ϕ_i' is selected from the distribution $p^{(0)}$; given the parameterisation of $p^{(0)}$ this process may be effected immediately through standard random number generators. (ii) The new variable is accepted or rejected according to the standard Metropolis algorithm, applied to the effective configurational energy $\mathscr{H}'$. This MC scheme was used to study the border model at a variety of couplings C, and for systems of linear dimension $L=8$, 16, 32 and 64, with periodic boundary conditions. The data upon which the following analysis are based were accumulated over approximately 80 h on the Edinburgh Distributed Array Processors.

The form of analysis chosen draws upon the ideas of Swendsen (1982), Binder (1981), Bruce (1981) and others. Its key ingredient is a finite size scaling ansatz for the probability distribution $P_L(M)$ of the block variable ('magnetisation') $M = L^{-d}\sum_i \phi_i$:

$$P_L(M) = L^{\beta/\nu}\tilde{p}(L^{\beta/\nu}M, L^{1/\nu}\mu_1, L^{-\omega}\mu_3) \tag{3}$$

where μ_1 and μ_3 denote, respectively, the thermal and leading irrelevant scaling fields. There are good reasons to believe (Bruce 1981) that, modulo the usual non-universal scale factors, the scaling function $\tilde{p}$ should, like the critical indices ($\beta, \nu, \omega, \ldots$), be specific to a universality class (though dependent upon the block boundary conditions: Binder (1981)).

The analysis falls into three parts. Firstly we address the location of the critical point, and the limiting critical form of the distribution P_L. The critical coupling C_c may be determined as that value of C for which the ratio of moments $M_L^{(n)} = \langle M_L^n\rangle$ of the PDF (3),

$$G_L = [3(M_L^{(2)})^2 - M_L^{(4)}]/2(M_L^{(2)})^2 \tag{4a}$$

attains, for large L, an L-independent fixed point value, G^*, intermediate between its high-temperature ($G_L = 0$) and low-temperature ($G_L = 1$) fixed point values. A series of relatively short MC runs at a variety of couplings yielded results establishing a preliminary estimate $C_c \simeq 0.3286$. A series of longer runs, upon which all subsequent analysis is based, were then performed at couplings in the vicinity of this preliminary estimate. The values of G_L thus obtained (shown in table 1) were analysed with the form

$$G_L = G^*[1 + g_1 L^{1/\nu}(C - C_c) + g_2 L^{-\omega} + \ldots] \tag{4b}$$

Table 1. Cumulant ratio G_L for various lattice sizes L and couplings C.

L	C	G_L
8	0.327 00	0.8758(5)
8	0.328 57	0.8852(4)
8	0.330 00	0.8945(4)
16	0.327 50	0.893(1)
16	0.328 57	0.905(1)
16	0.329 50	0.915(1)
32	0.328 00	0.903(3)
32	0.328 57	0.916(2)
32	0.329 00	0.924(4)
64	0.328 57	0.925(4)

which follows from (3), given its presumed analyticity at C_c (Binder 1981). (The consistency of the neglect of terms *quadratic* in $\Delta C \equiv C - C_c$, for the chosen C values, was confirmed with the aid of the studies of the C derivatives of the moments $M_L^{(n)}$, described below.) The key results are $C_c = 0.3282(2)$ and $G^* = 0.913(4)$ with a χ^2 of 1.5 (for four degrees of freedom). The errors (which are correlated) represent one standard deviation. The corrections to scaling will be discussed later. We also defer discussion of the index $1/\nu$; at this stage we note only that the assigned values of G^* and C_c are, within the quoted errors, independent of whether this index is assigned its Ising value or left as a free parameter. The value of C_c is significantly lower than that ($C_c = 0.3300$) assigned by BJ. The value of the fixed point moment ratio G^* may be compared with that of the $d = 2$ Ising model. A separate MC analysis of the latter yielded $G_I^* = 0.916(1)$. The close agreement between the two values is reflected more evocatively in the full probability distributions: figure 1 shows the critical point distributions for the two models, evaluated on an $L = 64$ lattice in each case. The accord is striking. The minor differences are presumably attributable to corrections to scaling: no attempt has been made to extrapolate to the $L = \infty$ limit.

We turn, secondly, to the index β/ν, which may be conveniently determined from the ratio of moments $R_L \equiv M_L^{(4)}/M_L^{(2)}$ for which we expect, given (3),

$$R_L = r_0 L^{-2\beta/\nu}[1 + r_1 L^{1/\nu}(C - C_c) + r_2 L^{-\omega} + \ldots]. \tag{5}$$

Fitting the observed border model values of R_L (for which the statistical noise is helpfully low) to this form gave $\beta/\nu = 0.128(^{+0.006}_{-0.003})$, and a critical coupling consistent with that assigned above, with a χ^2 value of 2 (for four DOF). Again the results (for the index and critical coupling) are, within error, insensitive to whether the index $1/\nu$ is fixed at its Ising value or left free. The quoted value of the index β/ν is consistent

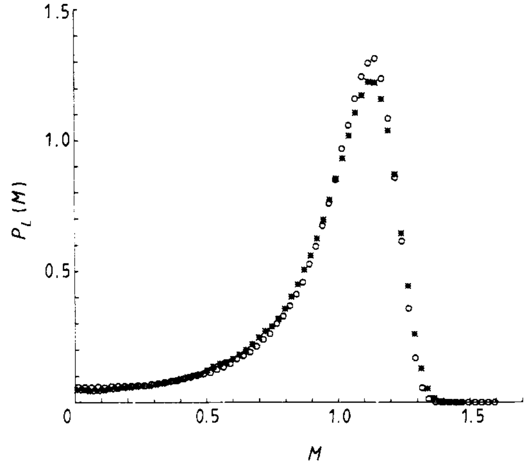

Figure 1. The magnetisation PDF for the $d=2$ border ϕ^4 model (*) and for the Ising model (○) at their respective critical points. Each calculation was performed on a system of size $L=64$. In each case the aggregate coordinate M has been scaled so that its variance is unity. (The functions are symmetric about $M=0$.)

with the Ising value $(\beta/\nu)_I = 0.125$, and markedly different from the value $\beta/\nu = \frac{1}{20}$ reported to be consistent with the BJ series values for γ and ν (Baker and Johnson 1985). This result, and the accord on the critical point distributions reported above, together offer a strong indication that the border ϕ^4 model does, in fact, fall into the same universality class as the Ising model, at least as regards those quantities which characterise the critical point coarse grained configurations of the ordering variable, so that universality holds at least in the weaker form described by Suzuki (1974).

To assess the claims of the strongest form of universality we turn, thirdly, to the index $1/\nu$. The values of this index suggested by the analyses of (4*b*) and (5) described above are markedly lower (0.83 and 0.92 respectively) than the Ising value ($1/\nu = 1$). Though statistically significant (the χ^2 values which ensue if this index is fixed at its Ising value are markedly higher than those for the free fits reported above) these discrepancies should not, we believe, be taken at face value: equations (4*b*) and (5) do not contain (and the fitting analysis would not support) corrections to the $L^{1/\nu}$ scaling behaviour presumably essential to a trustworthy assignment of this index. It is preferable, rather, to appeal to values of the *derivatives* $\mathrm{d}M_L^{(n)}/\mathrm{d}C$, determined directly with the aid of the identity

$$\frac{\mathrm{d}M_L^{(n)}}{\mathrm{d}C} = \left\langle \sum_{\langle ij \rangle} \phi_i \phi_j M^n \right\rangle_c \tag{6}$$

which follows immediately from equations (1*a*) and (1*b*). Monte Carlo measurements of the derivatives for $n=2$ and 4, determined in this way, were combined with those of the moments themselves to yield the quantity (chosen for its low statistical error)

$$S_L \equiv \frac{1}{M_L^{(4)}} \frac{\mathrm{d}M_L^{(4)}}{\mathrm{d}C} - \frac{1}{M_L^{(2)}} \frac{\mathrm{d}M_L^{(2)}}{\mathrm{d}C} \tag{7a}$$

for which equation (3) implies an expansion

$$S_L \simeq s_0 L^{1/\nu}[1 + s_1 L^{1/\nu}(C - C_c) + s_2 L^{-\omega} + \ldots]. \tag{7b}$$

Fitting with this form yielded the result $1/\nu = 1.01(3)$ in close accord with the Ising value. Our confidence in this result is, however, considerably less than the nominal error would suggest, for three reasons. Firstly the associated value of the critical coupling $C_c = 0.3275$ is somewhat lower than that indicated by the analyses already discussed: imposing the (previously determined) value $C_c = 0.3282$ yielded the result $1/\nu = 0.95(1)$, which is, in fact, consistent with BJ's most recently reported results (Baker and Johnson 1985). Secondly, in this analysis, in contrast to those discussed above, the corrections to scaling proved ill determined. The results cited were obtained with the index ω assigned the value suggested by the earlier analyses, to which we shall shortly return. Thirdly the quality of the fit (reflected in a χ^2 value of 5.05 for five DOF) is somewhat less satisfactory than that of the fitted representations of (4*b*) and (5). Evidently the situation as regards the index ν remains equivocal.

Finally, we consider the corrections to scaling entering equations (4*b*) and (5). In both instances the fitting analyses yielded a value for the index ω close to 1.3. This value lies suggestively close to that ($\omega = \frac{4}{3}$) which Nienhuis (1982) has suggested may be appropriate for the $d = 2$ Ising universality class (although the associated corrections to scaling need not always manifest themselves in the simple power law fashion suggested in (4*b*) and (5): cf Barma and Fisher 1984). It is noteworthy also that, for a given L, the size of the correction-to-scaling terms in equation (4*b*) is larger for the border model than it is for the Ising model by an order of magnitude (and, incidentally and understandably, of opposite sign): the scale of length (correlation length or system size) at which 'asymptotic' behaviour begins to be discernible must be correspondingly larger. This observation is simultaneously illuminating and disconcerting: it suggests that the troublesome features of the border model may indeed (as conjectured by Barma and Fisher 1984) lie in anomalously large corrections to scaling; at the same time it sets a worrying question mark (which the existing data do not allow us to dispel) beside the domain of acceptability of expansions such as (4*b*). Nevertheless, the balance of the evidence assembled here supports the view (at least as regards G^* and β/ν) that the universal characteristics of this asymptotic regime are indeed those of the Ising model.

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